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Zeros of Padé Approximants for Entire Functions with Smooth Maclaurin Coefficients

R. KOVACHEVA*

Institute of Mathematics with Comp. Center, Bulgarian Academy of Sciences, P.O. Box 373, 1090 Sofia, Bulgaria

AND

E. B. SAFF†

Department of Mathematics, Institute for Constructive Mathematics, University of South Florida, Tampa, Florida 33620

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For entire functions $f(z) = \sum_{j=0}^{\infty} a_j z^j$ whose coefficients satisfy the smoothness condition $a_{j+1}a_{j-1}/a_j^2 \to \eta$ as $j \to \infty$ we investigate the asymptotic behavior as $n \to \infty$ of the normalized partial sums $s_n(za_n/a_{n+1})$ and the normalized Padé numerators $P_{n,m}(za_n/a_{n+1})$, m fixed. As a consequence we deduce results on the limiting behavior of the zeros of these polynomials. © 1994 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

be an entire function in the complex plane $\mathbf{C}$ with $a_j \neq 0$ for all j ($j \in \mathbf{N}$) sufficiently large. We set

$$\eta_j := \frac{a_{j+1}a_{j-1}}{a_j^2}. \tag{1.1}$$

The basic assumption throughout the present work is that

$$\eta_j \to \eta \qquad \text{as } j \to \infty, \tag{1.2}$$

which we call the *Lubinsky smoothness condition* (see [1]).

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The function

$$H_\eta(z) := \sum_{j=0}^{\infty} \eta^{j(j+1)/2} z^j$$

plays an important role in our investigation. It is clear that for $|\eta| < 1$, $H_\eta(z)$ is an entire function. If $|\eta| = 1$, then H_η is holomorphic in the unit disk; in the case when $|\eta| > 1$, the radius of convergence is zero. We notice that $H_\eta(z) = h_\eta(\eta z)$, where $h_\eta(z)$ is the partial theta function.

We denote by $s_n(z) = s_n(f, z)$ the nth partial sum of the Maclaurin expansion for f:

$$s_n(z) := \sum_{j=0}^{n} a_j z^j.$$

Our first result describes the asymptotic behavior of normalized partial sums.

THEOREM 1.1. *Assume that* (1.2) *holds with* (i) $|\eta| < 1$ *or* (ii) $|\eta| = 1$ *and* $|\eta_j| \le 1$ *for all j large enough. Then, in case* (i),

$$s_n(ua_n/a_{n+1})/\{a_n(ua_n/a_{n+1})^n\} \to H_\eta(1/u) \tag{1.3}$$

as $n \to \infty$ *locally uniformly in* $\overline{\mathbf{C}} - \{0\}$ *and, in case* (ii),

$$s_n(ua_n/a_{n+1})/\{a_n(ua_n/a_{n+1})^n\} \to H_\eta(1/u) \tag{1.4}$$

locally uniformly in $\{u: |u| > 1\}$.

(As usual, "locally uniformly" means uniform convergence in the metric of Chebyshev on compact subsets.)

For each pair $(n, m) \in \mathbf{N}^2$, let $\pi_{n,m}$ $(= \pi_{n,m}(f))$ be the classical Padé approximant to the function f of type (n, m). Recall that $\pi_{n,m} = p/q$, $\deg p \le n$, $\deg q \le m$, $q \not\equiv 0$, where the polynomials p and q are determined by the condition

$$(fq - p)(z) = O(z^{n+m+1}) \qquad \text{as } z \to 0.$$

It is well known that for each pair (n, m) the function $\pi_{n,m}$ exists and is uniquely determined (see, e.g., [2]). We write

$$\pi_{n,m} = P_{n,m}/Q_{n,m},$$

where $Q_{n,m}(0) = 1$ and $P_{n,m}$ and $Q_{n,m}$ do not have a common divisor. Let

$$D(n,m) := \det(a_{n-j+k})_{j,k=1}^{m}, \qquad D(n,0) := 1,$$

be the Toeplitz determinant formed from the coefficients of the Maclaurin expansion for f. It is known (see, e.g., [3, 4]) that if m is fixed and f is not equal to a rational function having at most m poles in $\mathbf{C}$, then $D(n,m) \neq 0$ for an infinite sequence $\mathbf{N}' \subseteq \mathbf{N}$ and $\pi_{n,m} \equiv \pi_{k(n),m}$, where $k(n) := \max\{k: k \leq n,\ k \in \mathbf{N}'\}$. If f is rational with a denominator of degree $\leq m$, then $\pi_{n,m}(f) \equiv f$ for all $n \in \mathbf{N}$ sufficiently large. Without loss of generality, we shall assume hereafter that $\mathbf{N}' = \mathbf{N}$. In this case there holds (see [2])

$$(fQ_{n,m} - P_{n,m})(z) = \alpha_{n,m} z^{n+m+1} + \cdots,$$

with $\alpha_{n,m} \neq 0$ and

$$Q_{n,m}(z) = D(n,m)^{-1} \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} & a_{n+m} \\ a_{n-1} & a_n & \cdots & a_{n+m} & a_{n+m-1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-m+1} & a_{n-m+2} & \cdots & a_n & a_{n+1} \\ z^m & z^{m-1} & \cdots & z & 1 \end{vmatrix}. \tag{1.5}$$

Direct calculation also shows that

$$P_{n,m}(z) = z^n \frac{D(n,m+1)}{D(n,m)} + \cdots + d_{n,m}.$$

The next result supplies information about the behavior of the zeros of $\pi_{n,m}(f)$ as $n \to \infty$ in the case when the numbers η_n tend to η "smoothly enough"; namely, there exist complex numbers $\{c_i\}_1^\infty$ with $c_1 \neq 0$ such that for each positive integer $N > 1$,

$$\eta_j = \eta \cdot \left\{ 1 + c_1 j^{-1} + \sum_{i=2}^{N} c_i j^{-i} + o(j^{-N}) \right\} \qquad \text{as } j \to \infty. \tag{1.6}$$

This kind of convergence has been introduced and studied by D. S. Lubinsky in [1].

THEOREM 1.2. *Let the entire function f be given and $m \in \mathbf{N}$ be fixed. Assume that $a_j \neq 0$ for j large enough and that* (1.2) *holds with $\eta = 1$. Assume further that η_j admits the expansion* (1.6) *with $c_1 \neq 0$ and that*

$|\eta_j| \le 1$ *for all* j *sufficiently large. Then*

$$R_{n,m}(u) := \frac{P_{n,m}(ua_n/a_{n+1})}{(ua_n/a_{n+1})^n\{D(n,m+1)/D(n,m)\}} \to \frac{u^{m+1}}{(u-1)^{m+1}} \tag{1.7}$$

as $n \to \infty$ *locally uniformly in* $\{u\colon |u| > 1\}$.

Notice that for $m = 0$, we have $R_{n,m}(u) = s_n(ua_n/a_{n+1})/\{a_n(ua_n/a_{n+1})^n\}$ and the conclusion of Theorem 1.2 coincides with the conclusion of Theorem 1.1 in the special case when $\eta = 1$.

COROLLARY 1.3. *With the assumptions of Theorem* 1.2, *for each fixed* $m \in \mathbf{N}$ *and any* $\varepsilon > 0$, *the Padé approximant* $\pi_{n,m}(z)$ *has no zeros in* $|z| > |a_n/a_{n+1}|(1+\varepsilon)$ *for all* n *large.*

Set

$$R := \liminf_{n\to\infty} |a_n/a_{n+1}|.$$

Then, if $R < \infty$, there is an infinite sequence $\mathbf{N}'' \subset \mathbf{N}$ such that the zeros of $\pi_{n,m}(z)$, $n \in \mathbf{N}''$, have all their accumulation points in the closure of the disk $D_R = \{z\colon |z| < R\}$. On the other hand, it has been established in [1] that $\pi_{n,m}(z) \to f(z)$ locally uniformly inside D_R, so that any compact set in the interior of D_R contains not more than finitely many accumulation points of the zeros of $\{\pi_{n,m}\}_{n\to\infty}$. Consequently, all the zeros of $\pi_{n,m}$ with the exception of a finite number tend to $\partial D_R = \{z\colon |z| = R\}$ as $n \to \infty$, $n \in \mathbf{N}''$.

Examples of functions to which Theorem 1.2 may be applied are the exponential function

$$f(z) = e^z = \sum_{j=0}^{\infty} z^j/j!$$

(see [5]), the Mittag-Lefler function of order λ (> 0),

$$f(z) = \sum_{j=0}^{\infty} z^j/\Gamma(1 + j/\lambda)$$

(see [6]), as well as the function

$$f(z) = \sum_{j=0}^{\infty} (j!)^{-1/\lambda} z^j.$$

The next result provides more precise information concerning the limit points of the zeros of the sequence $\{\pi_{n,m}(f)\}$ as $n \to \infty$ for a special class of entire functions f.

THEOREM 1.4. *If $c_1 < 0$ in (1.6), then for each fixed $m \geq 0$, the point $u = 1$ is a limit point of zeros of the functions $\{R_{n,m}(u)\}_{n=1}^{\infty}$ in (1.7).*

COROLLARY 1.5. *Under the conditions of Theorem 1.4, for each fixed $m \in \mathbf{N}$ and any $\varepsilon > 0$, the Padé approximant $\pi_{n,m}(z)$ has at least one zero in the annulus*

$$\left|\frac{a_n}{a_{n+1}}\right|(1-\varepsilon) < |z| < \left|\frac{a_n}{a_{n+1}}\right|(1+\varepsilon)$$

for n large.

It follows from Corollary 1.5 that there is a sequence $\mathbf{L} \subset \mathbf{N}$ such that at least one zero of $\pi_{n,m}(z)$ tends to $z = \infty$ as $n \to \infty$, $n \in \mathbf{L}$, and the speed of the attraction is like $|a_n/a_{n+1}|$. If $\liminf_{n\to\infty} |a_n/a_{n+1}| = \infty$, then $\mathbf{L} = \mathbf{N}$.

Before we continue, we introduce the polynomials $B_m(u) = B_m(u, q)$, which are defined recursively as follows: $B_0(u) = 1$ and for $m = 1, 2, \ldots,$

$$B_m(u) = B_{m-1}(u) - uq^{m-1}B_{m-1}(u/q). \tag{1.8}$$

When q is not a root of unity, it can be shown that

$$B_m(-u) = \sum_{j=0}^{m} \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m+1-j})}{(1-q)(1-q^2)\cdots(1-q^j)}u^j;$$

furthermore, $B_m(u) = (1-u)^m$, when $q = 1$.

Details concerning the polynomials B_m, $m = 0, 1, 2, \ldots,$ can be found in [7]. These polynomials are of importance in the investigation of the distribution of the zeros of $\pi_{n,m}$ in the case when the number η in (1.2) is not a root of unity. The following theorem is valid:

THEOREM 1.6. *Assume that (1.2) holds for a number η that is not a root of unity. Then, for the Padé approximants $\pi_{n,m}$ associated with f there holds locally uniformly in $\mathbf{C} \setminus \mathscr{B}_m$,*

$$\lim_{n\to\infty} \{\pi_{n,m}(ua_n/a_{n+1}) - s_n(ua_n/a_{n+1})\}/\{a_n(ua_n/a_{n+1})^n\}$$
$$= -\sum_{k=0}^{m-1} \frac{(-u)^{k+1}\prod_{j=1}^{k}(1-\eta^j)}{B_k(u)B_{k+1}(u)}, \tag{1.9}$$

where $\mathscr{B}_m$ denotes the set of zeros of $B_1, \ldots, B_m$.

COROLLARY 1.7. *If condition* (i) *or* (ii) *of Theorem* 1.1 *holds and* η *is not a root of unity, then for the Padé numerators* $P_{n,m}$ *we have*

$$\lim_{n\to\infty} \frac{P_{n,m}(ua_n/a_{n+1})}{a_n(ua_n/a_{n+1})^n} = B_m(u)\left\{H_\eta\left(\frac{1}{u}\right) - \sum_{k=0}^{m-1} \frac{(-u)^{k+1}\prod_{j=1}^{k}(1-\eta^j)}{B_k(u)B_{k+1}(u)}\right\}$$

for u in the appropriate region described by Theorem 1.1.

2. PROOF OF THEOREM 1.1

Since the asymptotic results stated in Section 1 are not affected by the values of finitely many of the coefficients a_j, we assume for simplicity that $a_j \neq 0$ for all $j \geq 0$.

The proof of Theorem 1.1 requires the following simple lemma.

LEMMA 2.1. *For each pair of positive integers* (n, l) *such that* $n - l \geq 0$,

$$\frac{a_{n-l}}{a_n} = \left(\frac{a_{n+1}}{a_n}\right)^{-l} \prod_{k=1}^{l} \eta_{n+k-l}^{k}. \tag{2.1}$$

Proof. The proof is based on the equality

$$a_{n+i}a_n^{-1} = \eta_n a_n a_{n-i}^{-1}, \qquad i = \pm 1.$$

Let n be fixed. It is easy to verify that

$$a_{n-2}/a_n = (a_{n+1}/a_n)^{-2}\eta_{n-1}\eta_n^2.$$

Therefore, (2.1) is true for $l = 1$ and $l = 2$. Now suppose that (2.1) is valid for $l - 1$ and l with $3 \leq l < n$. We write

$$\begin{aligned} a_{n-l-1}/a_n &= (a_{n-l-1}/a_{n-l}) \cdot (a_{n-l}/a_n) \\ &= \eta_{n-l}(a_{n-l}/a_{n-l+1})(a_{n-l}/a_n) \\ &= \eta_{n-l}(a_{n-l}/a_n)^2(a_{n-l+1}/a_n)^{-1} \\ &= \eta_{n-l}(a_{n+1}/a_n)^{-2l}\prod_{k=1}^{l}\eta_{n-l+k}^{2k}\prod_{k=1}^{l-1}\eta_{n-l+1+k}^{-k}\cdot(a_{n+1}/a_n)^{l-1} \\ &= (a_{n+1}/a_n)^{-l-1}\eta_{n-l}\eta_{n-l+1}^2\prod_{k=2}^{l}\eta_{n-l+k}^{k+1} \end{aligned}$$

which yields (2.1) for $l + 1$, as required. ∎

For the proof of Theorem 1.1 we first set

$$Z_n(u) := s_n(ua_n/a_{n+1})/\{a_n(ua_n/a_{n+1})^n\} \tag{2.2}$$

and apply Lemma 2.1 to obtain

$$Z_n(u) = 1 + \sum_{j=1}^{n} u^{-j} \prod_{s=1}^{j} \eta_{n+s-j}^{s}. \tag{2.2}$$

Denote by $Z_{n,k}(u)$ the kth partial sum of $Z_n(u)$, $k \in \mathbf{N}$. We notice that for any fixed positive integer k, it follows from (1.2) that

$$Z_{n,k}(u) \to \sum_{j=0}^{k} u^{-j}\eta^{j(j+1)/2}, \qquad \text{as } n \to \infty, \tag{2.3}$$

locally uniformly in $\overline{\mathbf{C}} - \{0\}$. Since the limit polynomial in (2.3) is just the kth partial sum of the Laurent series for $H_\eta(1/u)$ about infinity, the proof of Theorem 1.1 will follow by estimating the function

$$\zeta_{n,k}(u) := Z_n(u) - Z_{n,k}(u) = \sum_{j=k+1}^{n} b_{n,j}/u^j,$$

where

$$b_{n,j} := \prod_{s=1}^{j} \eta_{n+s-j}^{s}. \tag{2.4}$$

Proof of Theorem 1.1(i). Suppose that $|\eta| < 1$. Let ε be a fixed positive number, $\varepsilon < 1$, and k_0 be a positive integer such that for all $k \geq k_0$ we have

$$\eta_k^* := \sup_{l \geq k} |\eta_l| < 1$$

and

$$(\eta_k^*)^{(k+1)/2} \leq \varepsilon/2. \tag{2.5}$$

Now let n be an integer with $n > 2k$. We write

$$\zeta_{n,k}(u) = \sum_{j=k+1}^{n-k+1} b_{n,j}/u^j + \sum_{j=n-k+2}^{n} b_{n,j}/u^j =: \zeta_{n,k}^{(1)}(u) + \zeta_{n,k}^{(2)}(u).$$

By (2.5), we have for $k \geq k_0$

$$\|\zeta_{n,k}^{(1)}\|_{\varepsilon \leq |u|} \leq \left((\eta_k^*)^{(k+1)/2}/\varepsilon\right)^k \sum_{j=k+1}^{n-k+1} \left((\eta_k^*)^{(j+k+1)/2}/\varepsilon\right)^{j-k}$$

$$\leq (1/2)^k \sum_{s=1}^{n-2k+1} (1/2)^s \leq (1/2)^k. \tag{2.6}$$

For $\zeta_{n,k}^{(2)}$ we obtain the following estimate for $k \geq k_0$,

$$\begin{aligned}\|\zeta_{n,k}^{(2)}(u)\|_{\varepsilon \leq |u|} &\leq \sum_{j=n-k+2}^{n} \varepsilon^{-j}\left(\prod_{s=1}^{k-n+j-1} |\eta_{n-j+s}^s| \prod_{s=k-n+j}^{j} |\eta_{n-j+s}^s|\right) \\ &\leq \sum_{j=2}^{k} \varepsilon^{-(j+n-k)}\left(\prod_{s=1}^{j-1} |\eta_{k-j+s}^s| \prod_{s=j}^{j+n-k} |\eta_{k-j+s}^s|\right) \\ &\leq \sum_{j=2}^{k} |\eta_1^*|^{(j-1)j/2} |\eta_k^*|^{(n-k+2j)(n-k+1)/2} / \varepsilon^{j+n-k} \\ &\leq C_1(k)\left(|\eta_k^*|^{(n-k+1)/2}/\varepsilon\right)^{n-k} \sum_{j=2}^{k} \left(|\eta_k^*|^{n-k+1}/\varepsilon\right)^j,\end{aligned}$$

where $C_1(k) := \max_{2 \leq j \leq k} |\eta_1^*|^{(j-1)j/2}$. Thus from (2.5), we see that for $n > 2k$

$$\|\zeta_{n,k}^{(2)}\|_{\varepsilon \leq |u|} \leq C_1(k)(1/2)^{n-k} \sum_{j=2}^{k} (1/2)^j \leq 2^k C_1(k)(1/2)^n.$$

Combining (2.6) and the last inequality, we deduce that

$$\|\zeta_{n,k}\|_{\varepsilon \leq |u|} \leq (1/2)^k + 2^k C_1(k)(1/2)^n$$

is valid for any positive integer $n > 2k$. Thus for a given $\delta > 0$ we can choose k so that for all n large enough we have

$$\|\zeta_{n,k}\|_{\varepsilon \leq |u|} < \delta.$$

Together with (2.3) this proves statement (i). ∎

Proof of Theorem 1.1(ii). Assume that

$$|\eta_j| \leq 1$$

for all $j > j_0$ and set $\lambda_0 := \max(1, \max_{j \leq j_0} |\eta_j|)$. Let τ be a fixed positive number. Then for $\zeta_{n,k}$ as defined in (2.4) we have

$$\|\zeta_{n,k}\|_{|u| \geq e^{\tau}} \leq \sum_{j=k+1}^{n-j_0} |b_{n,j}| e^{-j\tau} + \sum_{j=n-j_0+1}^{n} |b_{n,j}| e^{-j\tau}$$
$$\leq C\left(e^{-k\tau} + (\lambda_0)^{(j_0+1)j_0/2} \sum_{j=n-j_0+1}^{n} e^{-j\tau} \right),$$

where C is a constant independent of k and n. From here, statement (ii) follows easily. ▮

3. Preliminaries for the Proof of Theorem 1.2

In this section we state several lemmas that will be needed in the proof of Theorem 1.2. Since some of the new lemmas are quite technical, we relegate their proofs to the Appendix.

Lemma 3.1 (Lubinsky [1]). *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a formal power series, with $a_j \neq 0$ for j large enough.*

If (1.2) holds for a number η that is not a root of unity, then

$$\lim_{n \to \infty} D(n,m)/a_n^m = \prod_{j=1}^{m-1} (1 - \eta^j)^{m-j}$$

and

$$\lim_{n \to \infty} Q_{n,m}(ua_n/a_{n+1}) = B_m(u),$$

locally uniformly in $\mathbf{C}$.

Assume that $\eta_j = a_{j+1}a_{j-1}/a_j^2$ has the asymptotic expansion (1.6) with $c_1 \neq 0$ and $\eta = 1$. Then for $m = 1, 2, \ldots,$ we have as $n \to \infty$

$$D(n,m) = a_n^m(-c_1/n)^{m(m-1)/2} \prod_{j=1}^{m-1} j^{m-j} \cdot \left\{ 1 + \frac{\alpha(1,m)}{n} + o\left(\frac{1}{n}\right) \right\}.$$

Let $p \in \mathbf{N}$ be a fixed number and the function f be defined for $x > p$. We introduce the operator

$$\nabla^p f(x) := \sum_{k=0}^{p} \binom{p}{k} (-1)^k f(x-k)$$

with $\nabla^0 f(x) := f(x)$. Obviously,

$$\nabla^p f(x) = \nabla(\nabla^{p-1} f(x)) = \nabla^{p-1} f(x) - \nabla^{p-1} f(x-1).$$

LEMMA 3.2. *For each $p \in \mathbf{N}$, there holds*

$$\begin{aligned}\nabla^p(fg)(x) &= \sum_{k=0}^{p} \binom{p}{k} \nabla^k f(x) \nabla^{p-k} g(x-k) \\ &= \sum_{k=0}^{p} \binom{p}{k} \nabla^k g(x) \nabla^{p-k} f(x-k).\end{aligned} \tag{3.1}$$

Further, if $f^{(p)}(x)$ exists, then

$$\nabla^p f(x) = f^{(p)}(\xi_p), \tag{3.2}$$

for some $\xi_p \in (x-p, x)$.

The relations (3.1) and (3.2) are well-known facts from the theory of numerical methods (see, for example, [8]). From this lemma, it follows that

$$\begin{aligned}\nabla^p(f(x)^l) = \sum_{k_1=0}^{p} \sum_{k_2=0}^{p-k_1} \cdots \sum_{k_l=0}^{p-k_1-\cdots-k_{l-1}} \binom{p}{k_1} \nabla^{k_1} f(x) \\ \times \prod_{i=2}^{l} \nabla^{k_i} f\left(x - \sum_{j=1}^{i-1} k_j\right) \binom{p - \sum_{j=1}^{i-1} k_j}{k_i} \nabla^{p-\sum_{j=1}^{l} k_j} f\left(x - \sum_{j=1}^{l} k_j\right).\end{aligned} \tag{3.3}$$

LEMMA 3.3. *Assume that η_j is of the form* (1.6) *with $\eta = 1$ and $c_1 \neq 0$. Let N be a fixed positive integer.*

(a) *Then for each $j < n/(N+1)$ we have*

$$\prod_{l=1}^{j} \eta_{n-l} = 1 + \sum_{s=1}^{N} j\mathscr{P}_{s-1}(j)/n^s + M_{N+1}(j,n), \qquad n \to \infty, \tag{3.4}$$

where $\mathscr{P}_s$, $s = 1, \ldots, N-1$, are polynomials that do not depend on n but only on the coefficients c_s, $s = 1, \ldots, N-1$; the degree of each $\mathscr{P}_s$ does not exceed s; there is a constant $C_1(N+1)$ that depends only on N and c_s, $s = 1, \ldots, N$, such that

$$n^{N+1} |M_{N+1}(j,n)| \leq C_1(N+1) j^{N+1}. \tag{3.5}$$

(b) *For any positive integer p and for $j < (n-p)/(N+1+2p)$ there holds*

$$\left|n^{N+p+1}\nabla^p M_{N+1}(j,n)\right| \le C_2(N+1,p)j^{N+1}, \qquad \text{as } n \to \infty, \quad (3.6)$$

where $C_2(N+1,p)$ is a constant that does not depend on j and n.

From Lemma 3.3, we obtain

LEMMA 3.4. *Let $l, p, N \in \mathbf{N}$ be fixed. Then, for $j < (n-p)/(N+2p)$ we have*

$$\left|n^{N+p}\nabla^p\{M_N(j,n)/n^l\}\right| \le C_3(l,p,N)\cdot j^N, \qquad n \to \infty,$$

where $C_3(l,p,N)$ is a suitable positive constant.

LEMMA 3.5. *Let $p, N \in \mathbf{N}$ be fixed. Then, for $j < (n-p)/(N+2p)$ there holds*

$$\left|n^{N+p-1}\nabla^p(n\cdot M_N(j,n))\right| \le C_4(p,N)\cdot j^N, \qquad n \to \infty.$$

We also need the following.

LEMMA 3.6. *Let $p, N \in \mathbf{N}$ be fixed. Then, for $j < (n-1-p)/(N+2(p+1))$ we have*

$$\left|n^{N+p}\nabla^p\{n\nabla M_N(j,n)\}\right| \le C_5(p,N)j^N, \qquad n \to \infty.$$

Remark 1. Lemma 3.3 may be applied to any product $\prod_{l=l_0}^{j}\eta_{n-l}$ for $0 \le l_0 < j$ and j sufficiently "small." Indeed,

$$\prod_{l=l_0}^{j}\eta_{n-l} = \prod_{l=1}^{j+1-l_0}\eta_{(n-l_0+1)-l},$$

and so Lemma 3.3 applies on replacing n by $n-l_0+1$ and j by $j+1-l_0$.

In the special case when $l_0 = 0$, we obtain for n sufficiently large and for $j+1 < (n+1)/(N+1)$ that

$$\prod_{l=0}^{j}\eta_{n-l} = 1 + c_1(j+1)/n + \sum_{s=2}^{N}(j+1)\mathscr{P}_{s-1}(j+1)/(n+1)^s$$
$$+ M_{N+1}(j+1,n+1).$$

We rewrite the last equality as

$$\prod_{l=0}^{j} \eta_{n-l} = 1 + \sum_{s=1}^{N} (j+1)\mathscr{C}_{s-1}(j)/n^s + \mathscr{N}_{N+1}(j,n). \tag{3.7}$$

Arguing as in the proof of Lemma 3.3, we easily establish that $\deg \mathscr{C}_s \le s$; moreover, for $j + 1 < (n+1)/(N+1)$ there holds

$$\left|n^{N+1}\mathscr{N}_{N+1}(j,n)\right| \le j^{N+1}C_6(N+1), \qquad n \to \infty; \tag{3.8}$$

and for $j + 1 < (n + 1 - p)/(N + 1 + 2p)$ we have

$$\left|n^{N+p+1}\mathscr{N}_{N+1}(j,n)\right| \le j^{N+1}C_7(p, N+1), \qquad n \to \infty. \tag{3.9}$$

We notice that

$$\mathscr{C}_0(j) = c_1 \qquad \text{and} \qquad \mathscr{C}_1(j) = \left(jc_1^2 + jc_1 + 2c_2\right)/2. \tag{3.10}$$

LEMMA 3.7. *Let n and m be fixed positive integers. Then there holds (for $D(n,m) \neq 0$)*

$$\pi_{n,m+1}(z) - \pi_{n,m}(z) = (-1)^m \frac{D(n+1, m+1) \cdot z^{n+m+1}}{D(n,m) \cdot Q_{n,m}(z) \cdot Q_{n,m+1}(z)}. \tag{3.11}$$

4. PROOF OF THEOREM 1.2

Let m be a fixed positive integer. We recall that $D(n,m) = (a_{n+j-i})_{i,j=1}^m$ is the Toeplitz determinant of order m. We shall assume that $D(n,m) \neq 0$ for each $n \in \mathbf{N}$. We set

$$Q_{n,m}(z) = \sum_{k=0}^{m} q_{k,n,m} z^{m-k}, \tag{4.1}$$

where $q_{k,n,m}$ is given by the expression

$$q_{k,n,m} = \frac{(-1)^{m-k}}{D(n,m)} \times \begin{vmatrix} a_n & \cdots & a_{n+k-1} & a_{n+k+1} & \cdots & a_{n+m} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-m+1} & \cdots & a_{n-m+k} & a_{n-m+k+2} & \cdots & a_{n+1} \end{vmatrix}$$

(notice that $q_{m,n,m} = 1$).

It is known (see [4]) that

$$P_{n,m}(z) = D(n,m)^{-1}$$
$$\times \begin{vmatrix} a_n & \cdots & a_{n+k} & \cdots & a_{n+m} \\ \vdots & & \vdots & & \vdots \\ a_{n-m+1} & \cdots & a_{n-m+k+1} & \cdots & a_{n+1} \\ \sum_{j=m}^{n} a_{j-m} z^j & \cdots & \sum_{j=m-k}^{n} a_{j-m+k} z^j & \cdots & \sum_{j=0}^{n} a_j z^j \end{vmatrix}.$$

In view of (4.1) we may write

$$P_{n,m}(z) = \sum_{k=0}^{m} q_{k,n,m} \left\{ \sum_{j=m-k}^{n} a_{j-m+k} z^j \right\}.$$

This yields

$$\begin{aligned} P_{n,m}(z) &= \sum_{k=0}^{m} q_{k,n,m} z^{m-k} s_{n-m+k}(z) \\ &= s_n(z) Q_{n,m}(z) - \sum_{k=0}^{m-1} q_{k,n,m} z^{m-k} \{ s_n(z) - s_{n-m+k}(z) \}. \end{aligned}$$

From this, we get

$$\begin{aligned} R_{n,m}(u) &= \{P_{n,m}(ua_n/a_{n+1})\} / \{D(n,m+1)(ua_n/a_{n+1})^n / D(n,m)\} \\ &= \left(\frac{s_n(ua_n/a_{n+1})}{a_n(ua_n/a_{n+1})^n} \right) Q_{n,m}(ua_n/a_{n+1}) a_n D(n,m)/D(n,m+1) \\ &\quad - \frac{a_n D(n,m)}{D(n,m+1)} \sum_{k=0}^{m-1} q_{k,n,m} (ua_n/a_{n+1})^{m-k} \\ &\quad \times \left\{ \frac{(s_n - s_{n-m+k})(ua_n/a_{n+1})}{a_n (ua_n/a_{n+1})^n} \right\}. \end{aligned}$$

Now by (2.2), we may write

$$R_{n,m}(u) = \{ Z_n(u) Q_{n,m}(ua_n/a_{n+1}) a_n D(n,m)/D(n,m+1) \}^*,$$

where $\{\cdots\}^*$ denotes the sum of terms with nonpositive powers of u in $\{\cdots\}$. Applying (2.2)′ and (4.1), we obtain the formula

$$R_{n,m}(u) = 1 + \sum_{j=1}^{n} \left(b_{n,j}/u^j\right) A_{n,j,m}, \tag{4.2}$$

where $b_{n,j}$ is given by (2.4), and

$$A_{n,j,m} = \begin{cases} \left(\dfrac{a_n D(n,m)}{D(n,m+1)}\right) \displaystyle\sum_{k=0}^{m} q_{m-k,n,m} \left(\dfrac{a_n}{a_{n+1}}\right)^k \prod_{i=1}^{k} \eta_{n-k-j+i}^{i} \prod_{i=1}^{j} \eta_{n+i-j}^{k}, \\ \qquad\qquad\qquad\qquad\qquad \text{for } j = 1, \ldots, n-m \\ \left(\dfrac{a_n D(n,m)}{D(n,m+1)}\right) \displaystyle\sum_{s=0}^{n-j} \prod_{r=1}^{j+s} \eta_{n-j-s+r} q_{m-s,n,m} (a_n/a_{n+1})^s, \\ \qquad\qquad\qquad\qquad\qquad \text{for } j = n-m+1, \ldots, n \end{cases} \tag{4.3}$$

(we set $\prod_{i=1}^{k} \cdots = 1$ for $k = 0$).

It is known (see [3, 4]) that for each $m \in \mathbf{N}$

$$Q_{n,m}(z) = Q_{n,m-1}(z) - zQ_{n-1,m-1}(z)\hat{D}(n,m),$$

where

$$\hat{D}(n,m) := \{D(n-1,m-1)D(n+1,m)\}/\{D(n,m-1)D(n,m)\}.$$

From this, we obtain

$$q_{m-k,n,m} = \begin{cases} q_{m-k-1,n,m-1} - \hat{D}(n,m) q_{m-k,n-1,m-1}, & \text{for } 1 \le k \le m-1 \\ -q_{0,n-1,m-1}\hat{D}(n,m) & \text{for } k = m. \end{cases}$$

Using these formulas, (4.3), and also the equality $a_n \eta_n / a_{n+1} = a_{n-1}/a_n$, we obtain for $j \le n-m$

$$\begin{aligned} A_{n,j,m} = {} & \left\{D(n,m)^2/(D(n,m+1)D(n,m-1))\right\} A_{n,j,m-1} \\ & - \{D(n+1,m)D(n-1,m)/(D(n,m+1)D(n,m-1))\} \\ & \times A_{n-1,j,m-1} \prod_{i=1}^{j} \eta_{n-i}. \end{aligned} \tag{4.4}$$

Now let m be a fixed positive integer. We shall prove that for any positive integer N and for every $j < n/(N + 3m - 1)$ the following expansion is valid as $n \to \infty$,

$$A_{n,j,m} = \left\{\prod_{i=1}^{m}(j+i)\right\}/m! + \sum_{s=1}^{N-1} T_{m,s}(j)/n^s + \mathcal{F}_{N,m}(j,n), \quad (4.5)$$

where $T_{m,s}$ are polynomials of degree $\leq m + s$, $s = 1, \dots, N - 1$, and

$$\left|n^N \mathcal{F}_{N,m}(j,n)\right| \leq c(m,N) j^{N+m}, \qquad n \to \infty, \quad (4.6)$$

with the constant $c(m, N)$ not depending on j or n. (We set $T_{m,s} := 0$ for $N = 1$.) Also, for any $p \in \mathbf{N}$ and for $j < n/(N + m + 1 + 2(p + m - 1))$ we shall prove that

$$\left|n^{N+p} \nabla^p(\mathcal{F}_{N,m}(j,n))\right| \leq c(m,N,p) j^{N+m}, \qquad n \to \infty \quad (4.7)$$

for a suitable positive constant $c(m, N, p)$, where ∇^p is defined in Section 3. Furthermore we will prove that

$$|A_{n,j,m}| \leq c_2(m) n^{m+1}, \qquad n \to \infty \quad (4.8)$$

for $j \geq n/3m$.

From (4.5) and (4.6), for $N = 1$ and $j < n/3m$, it follows that

$$A_{n,j,m} = \left\{\prod_{i=1}^{m}(j+i)\right\}/m! + \mathcal{F}_{1,m}(j,n), \qquad n \to \infty, \quad (4.9)$$

where

$$\left|n \mathcal{F}_{1,m}(j,n)\right| \leq c_1(m) j^{m+1} \quad (4.10)$$

(here $c_1(m) = c(m, 1)$).

The proofs of (4.5)–(4.8) are given in the Appendix.

Now we are in the position to prove Theorem 1.2. The proof repeats the ideas of the proof of Theorem 1.1. For any fixed $j \in \mathbf{N}$, we obtain that

$$A_{n,j,m} b_{n,j} \to \frac{1}{m!} \prod_{i=1}^{m}(j+i), \qquad \text{as } n \to \infty. \quad (4.11)$$

Let δ be an arbitrary positive number and k be a positive integer. From

(4.2), we may write

$$\|R_{n,m}\|_{|u|>e^{\delta}} \leq 1 + \sum_{j=1}^{k} A_{n,j,m} b_{n,j} e^{-\delta j} + \sum_{j=k+1}^{n/3m-1} A_{n,j,m} b_{n,j} e^{-\delta j} + \sum_{j=n/3m}^{n} A_{n,j,m} b_{n,j} e^{-\delta j} = 1 + \zeta_{n,m,k} + \zeta'_{n,m,k} + \zeta''_{n,m,k}.$$

We shall now estimate the last two terms.

Using (4.9), (4.10), and the fact that $|b_{n,j}| \leq C_8$ for all n and j, where C_8 is a suitable constant, we establish that

$$\zeta'_{n,m,k} \leq c e^{-\delta k/2}, \qquad k \geq k_0. \tag{4.12}$$

From (4.8) we obtain that

$$\zeta''_{n,m,k} = o(1/n), \qquad n \to \infty.$$

Combining this last result, (4.11), and (4.12), and using the fact that δ is arbitrary, we conclude that

$$R_{n,m}(u) \to 1 + \sum_{j=1}^{\infty} \left\{ \frac{1}{m!} \prod_{i=1}^{m} (j+i) \right\} \frac{1}{u^j} \tag{4.13}$$

uniformly inside $\mathbf{C} \setminus \{u\colon |u| \leq 1\}$.

It is not difficult to show that for $|u| > 1$

$$1 + \sum_{j=1}^{\infty} \left\{ \frac{1}{m!} \prod_{i=1}^{m} (j+i) \right\} \frac{1}{u^j} = \frac{u^{m+1}}{(u-1)^{m+1}}.$$

Indeed,

$$\begin{aligned}
\sum_{j=0}^{\infty} \left\{ \frac{1}{m!} \prod_{i=1}^{m} (j+i) \right\} \frac{1}{u^j} &= \frac{u^{m+1}}{m!} \sum_{j=0}^{\infty} \left\{ \prod_{i=1}^{m} (j+i) \right\} \frac{1}{u^{j+m+1}} \\
&= (-1)^m \frac{u^{m+1}}{m!} \sum_{j=0}^{\infty} \left(\frac{1}{u^{j+1}} \right)^{(m)} \\
&= (-1)^m \frac{u^{m+1}}{m!} \left(\frac{1}{u} \frac{u}{u-1} \right)^{(m)} \\
&= \frac{(-1)^m u^{m+1}}{m!} \frac{(-1)^m m!}{(u-1)^{m+1}} = \frac{u^{m+1}}{(u-1)^{m+1}}.
\end{aligned}$$

If $m = 0$, then

$$\frac{u^{m+1}}{(u-1)^{m+1}} = \frac{u}{u-1} = \frac{1}{1-1/u} = H_1\left(\frac{1}{u}\right).$$

This proves Theorem 1.2. ∎

5. Proof of Theorem 1.4

Let $m \in \mathbf{N}$ be fixed. Recall that

$$R_{n,m}(u) = 1 + \sum_{j=1}^{n} A_{n,j,m} b_{n,j} u^{-j},$$

where the behavior of $A_{n,j,m}$ as $n \to \infty$ is described by (4.8)–(4.10) and $b_{n,j}$, $j = 1, \ldots, n$, are given from (2.4) by the formula

$$b_{n,j} = \prod_{l=1}^{j} \eta_{n-j+l}^{l}.$$

Hereafter, we shall write $R_{n,m} = R_n$.

Let ε be a fixed positive number, $\varepsilon < 1$, and determine n_1 from (A.1) of the Appendix. We shall use in our further considerations the notation

$$\eta_n = 1 - 2d_1/n + o(1/n) = 1 - 2d_1/n + c_2/n^2 + o(1/n^2), \quad (5.1)$$

where $d_1 := -c_1/2 > 0$. In accordance with the previous notations $|o(1/n)| < \varepsilon$ and $n^2|o(1/n^2)| \leq C'(1)$ for an appropriate positive constant $C'(1)$ (see (A.2)). With respect to the conditions of the theorem, we shall assume that

$$|\eta_n| \leq 1 - d_1/n \quad (5.2)$$

for $n > n'(> n_1)$.

Suppose the statement of Theorem 1.4 is not true. Then there exists a disk $\mathscr{U}$ containing $u = 1$ such that $R_n(u) \neq 0$ for every $u \in \mathscr{U}$ and for n sufficiently large ($n > n'' > n'$). Assuming that the radius of $\mathscr{U}$ is less than 1, we fix a simply connected domain $\mathscr{V}, \mathscr{V} \subset \mathscr{U} \cup \{u\colon |u| > 1\}$ with $1, \infty \in \mathscr{V}$. Let X_n, $n > n'$ be the regular branch in $\mathscr{V}$ of the function $R_n(u)^{1/n}$, determined by the condition $R_n(\infty)^{1/n} = 1$. The functions $X_n(u)$ are holomorphic (analytic and single-valued) in $\mathscr{V}$ and, as it is easy to establish (see, for example, [9]), uniformly bounded there. Hence $\{X_n\}$ forms a normal family in $\mathscr{V}$. On the other hand, Theorem 1.2 implies that

$X_n(u) \to 1$ as $n \to \infty$ locally uniformly inside $\{u\colon |u| > 1\}$. Thus, by the theorem of uniqueness for holomorphic functions

$$X_n(u) \to 1, \qquad \text{as } n \to \infty, \tag{5.3}$$

locally uniformly throughout $\mathscr{Y}$.

Now select a positive number δ_0 such that $u = e^{-2\delta_0} \in \mathscr{U}$ and consider

$$R_n(e^{-2\delta}) = 1 + \left\{ \sum_{j=1}^{n} A_{n,j,m} b_{n,j} e^{2\delta j} \right\}$$

for $\delta < \delta_0$.

We shall show that for each δ sufficiently small $\operatorname{Re}\{R_n(e^{-2\delta})\}$ increases as $n \to \infty$ with a speed $\geq e^{c\delta n}$, where c is a suitable positive constant. This contradicts (5.3).

It follows from (1.6) that there exists a positive integer n_0, $n_0 \geq n''$ such that for $n > n_0$ the inequality

$$|\eta_n| \leq |\eta_{n+1}| \tag{5.4}$$

holds. Indeed, from the inequality $2d_1 > 0$ it easily follows that

$$|\eta_n| = 1 - \frac{2d_1}{n} + \frac{\lambda}{n^2} + o\left(\frac{1}{n^2}\right),$$

which yields (5.4).

Further, we assume also that for $n > n_0$ the following inequalities are satisfied:

$$(n/d_1)|\log(1 - d_1/n)| \geq 1 - \varepsilon, \tag{5.5}$$

and

$$|\operatorname{Im} \eta_n| \leq \mathscr{C}(1) \cdot \operatorname{Re} \eta_n / n^2 \tag{5.6}$$

for a suitable positive constant $\mathscr{C}(1)$. With respect to (4.10) and (4.8) we may assume also that for $n > n_0$

$$|\mathscr{F}_{1,m}(j,n)| \leq c_1(m) j^{m+1}/n \tag{5.7}$$

for $j < n/3m$ and

$$|A_{n,j,m}| \leq c_2(m) n^{m+1} \tag{5.8}$$

for $j \geq n/3m$. We assume, also, without loss of generality, that $c_1(m) > 1$.

Now, using (5.4) and (5.2), we get

$$|b_{n,j}| \le |\eta_n|^{j(j+1)/2} \le (1 - d_1/n)^{j(j+1)/2} \tag{5.9}$$

for $j < n - n_0 + 1$ and

$$|b_{n,j}| \le |\eta^*_{n_0}|^{n_0(n_0+1)/2}(1 - d_1/n)^{(n-n_0)(2j+n_0-n)/2} \tag{5.10}$$

for $j \ge n - n_0 + 1$; here $\eta^*_{n_0} = \max_{n \le n_0} |\eta_n|$.

Now set $\delta_1 = \min\{\delta_0, d_1(1-\varepsilon)/\{6m!(c_1(m)+1)\}\}$. We introduce the functions

$$\varphi_1(\delta) := \mathscr{C}(1)\{6\delta/(d_1(1-\varepsilon)) - 1\}^{-2},$$
$$\varphi_2(\delta) := (\varphi_1(\delta)/2)(6/d_1(1-\varepsilon))^2.$$

Select the positive number δ, $\delta < \delta_1$, such that

$$\delta^2\varphi_2(\delta) < 1/2 \quad \text{and} \quad 1/m! - 6c_1(m)\delta/(d_1(1-\varepsilon)) > \delta^2\varphi_2(\delta)(>0)$$

and the number $\delta/d_1(1-\varepsilon)$ is irrational.

Without loss of generality, in our further considerations we shall assume for $n > n_0$ that the following additional inequalities are fulfilled:

$$\operatorname{Re}\eta_n \ge 1 - (2d_1 + \delta)/n > 0 \tag{5.11}$$

and

$$n \cdot \log\{1 - (2d_1+\delta)/\{n\{1 - 6\delta/d_1(1-\varepsilon)\} + 2\}\}$$
$$\ge -2(d_1+\delta)/\{1 - 6\delta/(d_1(1-\varepsilon))\}. \tag{5.12}$$

Now, we easily obtain that for $n > n_0$ and for every positive number j, satisfying the inequalities $(6n\delta)/(d_1(1-\varepsilon)) - 1 < j < n - n_0 + 1$, there holds

$$|b_{n,j}| \le e^{-3\delta j}. \tag{5.13}$$

Indeed, from (5.5) we obtain

$$(j+1)|\log(1 - d_1/n)|/2 \ge 3\delta$$

and hence

$$(1 - d_1/n)^{(j+1)/2} \le e^{-3\delta}.$$

Combining this inequality and (5.9), we obtain (5.13).

Further, from (4.9), (5.13), (5.7), and (5.8) we see that

$$\left|\sum_{j=6\delta n/d_1(1-\varepsilon)}^{n-n_0} A_{n,j,m} b_{n,j} e^{2\delta j}\right| \leq c^*(m) n^{m+1} \sum_{j=6\delta n/d_1(1-\varepsilon)}^{\infty} e^{-j\delta} \leq \mathscr{C}(2), \tag{5.14}$$

where $c^*(m) = \max(c_1(m), c_2(m))$ and $\mathscr{C}(2)$ is a suitable constant.

On the other hand, in light of (4.10) and (5.10) we may write

$$\left|\sum_{j=n-n_0+1}^{n} A_{n,j,m} b_{n,j} e^{2\delta j}\right|$$
$$< c^{**}(m) n^{m+1} \sum_{j=n-n_0+1}^{n} \{(1-d_1/n)\}^{j+(n-n_0)(n-n_0+1)/2} e^{2\delta j},$$

where

$$c^{**}(m) := c_2(m) |\eta^*_{n_0}|^{n_0(n_0+1)/2}.$$

Therefore, there holds for n sufficiently large

$$\left|\sum_{j=n-n_0+1}^{n} A_{n,j,m} b_{n,j} e^{2\delta j}\right| \leq \mathscr{C}(3). \tag{5.15}$$

Now let $j \leq 6\delta n/\{d_1(1-\varepsilon)\} - 1$. In view of (4.10) and of the choice of δ, there holds

$$|\mathscr{F}_{1,m}(j,n)| \leq \mathscr{C}(4)\delta j^m,$$

where $\mathscr{C}(4) = c_1(m)6/(d_1(1-\varepsilon))$. Hence

$$|\operatorname{Im} A_{n,j,m}| \leq \mathscr{C}(4)\delta j^m. \tag{5.16}$$

and

$$\operatorname{Re} A_{n,j,m} > j^m\{1/m! - \mathscr{C}(4)\delta\} + \left\{\sum_{l=1}^{m} (j+l) - j^m\right\}/m!. \tag{5.17}$$

Denote by $\mathscr{P}(j)$ the polynomial on the right-hand side in the last inequality. In virtue of the choice of δ, it follows that the degree of $\mathscr{P}(j)$ is exactly m and all its coefficients are positive. Therefore we may write, for $1 < j \leq 6\delta n\{d_1(1-\varepsilon)\} - 1$

$$\operatorname{Re} A_{n,j,m} \geq \mathscr{P}(j) > 0. \tag{5.18}$$

Now, consider

$$b_{n,j}/\prod_{l=1}^{j}\{\operatorname{Re}\eta_{n-j+l}\}^{l} = \prod_{l=1}^{j}\{1 + i(\operatorname{Im}\eta_{n-j+l})/(\operatorname{Re}\eta_{n-j+l})\}^{l}$$
$$= \prod_{l=1}^{j}\{1 + \mu_{n-j+l}\}^{l},$$

where

$$\mu_{n-j+l} = i(\operatorname{Im}\eta_{n-j+l})/(\operatorname{Re}\eta_{nj+l}).$$

From (5.6), we get for $j \le 6\delta n/\{d_1(1-\varepsilon)\} - 1$, for $l = 1,\ldots,j$ and for $n > n_0' \ge n_0/\{1 - 6\delta/(d_1(1-\varepsilon))\}$

$$|\mu_{n-j+l}| \le \mathscr{C}(5)/n^2, \tag{5.19}$$

where $\mathscr{C}(5) = \varphi_1(\delta) = \mathscr{C}(1)\{6\delta/(d_1(1-\varepsilon)) - 1\}^{-2}$.

Set

$$x_l := \mu_{n-j+l},\ l = 1,\ldots,j \qquad \text{and} \qquad F(x_1,\ldots,x_j) := \prod_{l=1}^{j}(1+x_l)^{l}.$$

By the choice of j and n, (5.19) implies that

$$|F(x_1,\ldots,x_j) - 1| \le \left|\sum_{l=1}^{j} F_{x_l}(0,\ldots,0)x_l\right|$$
$$= \left|\sum_{l=1}^{j} lx_j\right| \le \frac{\mathscr{C}(5)}{n^2}\frac{(j+1)}{2}j \le \mathscr{C}(6)\delta^2,$$

where

$$\mathscr{C}(6) := \frac{\mathscr{C}(5)}{2}\left(\frac{6}{d_1(1-\varepsilon)}\right)^2 = \varphi_2(\delta).$$

The choice of δ implies

$$\left|\prod_{l=1}^{j}(1+\mu_{n-j+l})^{l} - 1\right| \le \varphi_2(\delta)\delta^2; \tag{5.20}$$

thus we obtain

$$\left|\operatorname{Re}\left\{b_{n,j}/\prod_{l=1}^{j}\{\operatorname{Re}\eta_{n-j+l}\}^{l} - 1\right\}\right| \le \varphi_2(\delta)\delta^2$$

and also

$$\left|\operatorname{Im}\left\{b_{n,j}/\prod_{l=1}^{j}\{\operatorname{Re}\eta_{n-j+l}\}^{l}-1\right\}\right|\leq\varphi_2(\delta)\delta^2.$$

Consequently,

$$\begin{aligned}(1-\varphi_2(\delta)\delta^2)\prod_{l=1}^{j}\{\operatorname{Re}\eta_{n-j+l}\}^{l}&\leq\operatorname{Re}b_{n,j}\\&\leq\prod_{l=1}^{j}\{\operatorname{Re}\eta_{n-j+l}\}^{l}(1+\varphi_2(\delta))\end{aligned}\tag{5.21}$$

and

$$|\operatorname{Im}b_{n,j}|\leq\varphi_2(\delta)\delta^2\prod_{l=1}^{j}\{\operatorname{Re}\eta_{n-j+l}\}^{l}.\tag{5.22}$$

On the other hand, the choice of δ and (5.11) yields

$$\begin{aligned}&\prod_{l=1}^{j}\{\operatorname{Re}\eta_{n-j+l}\}^{l}\\&\qquad\geq\left\{1-(2d_1+\delta)/\left\{n\{1-6\delta/d_1(1-\varepsilon)\}+2\right\}\right\}^{j(j+1)/2}>0.\end{aligned}\tag{5.23}$$

Finally, from (5.18), (5.21), (5.16), and (5.22) we obtain

$$\begin{aligned}&\operatorname{Re}A_{n,j,m}\operatorname{Re}b_{n,j}-\operatorname{Im}A_{n,j,m}\operatorname{Im}b_{n,j}\\&\quad\geq Q(j)\prod_{l=1}^{j}\{\operatorname{Re}\eta_{n-j+l}\}^{l}\\&\quad\geq Q(j)\left\{1-(2d_1+\delta)/\left\{n\{1-6\delta/d_1(1-\varepsilon)\}+2\right\}\right\}^{j(j+1)/2},\end{aligned}\tag{5.24}$$

where, in view of (5.17),

$$\begin{aligned}Q(j)&=(1-\varphi_2(\delta)\delta^2)\mathscr{P}(j)-\mathscr{C}(4)\varphi_2(\delta)\delta^3j^m\\&=j^m\left\{\frac{1}{m!}-\frac{c_1(m)6\delta}{d_1(1-\varepsilon)}-\varphi_2(\delta)\delta^2\right\}+\cdots.\end{aligned}$$

The choice of δ ensures that the degree of the polynomial $Q(j)$ is exactly

m and all its coefficients are positive. Combining this result and (5.11) we obtain

$$\operatorname{Re} A_{n,j,m} \operatorname{Re} b_{n,j} - \operatorname{Im} A_{n,j,m} \operatorname{Im} b_{n,j} > 0. \tag{5.25}$$

Select a positive number α such that $\alpha < 2\delta$ and set $\kappa := (2\delta - \alpha)\{1 - 6\delta/d_1(1 - \varepsilon)\}/(d_1 + \delta)$. Consider

$$\mathscr{A}(\delta, n, \alpha) = \operatorname{Re} \sum_{j=1}^{[\kappa n]-1} A_{n,j,m} b_{n,j} e^{2\delta j}.$$

It is easy to establish that for $j \le \kappa n - 1$ the inequality

$$\prod_{l=1}^{j} \{\operatorname{Re} \eta_{n-j+l}\}^{l} > e^{(\alpha - 2\delta)j} \tag{5.26}$$

is valid. Indeed, from the choice of κ we get

$$-\{(j+1)/2n\}\{2(d_1 + \delta)/\{1 - 6\delta/d_1(1 - \varepsilon)\}\} + 2\delta > \alpha.$$

Applying (5.12) we see that for n sufficiently large

$$\{(j+1)/2\}\log\{1 - (2d_1 + \delta)/\{n(1 - 6\delta/d_1(1 - \varepsilon) + 2\}\} + 2\delta > \alpha.$$

Inequality (5.26) now follows from (5.23) and from the fact that $n > n_0'$.
Combining now (5.14), (5.15), (5.24), and (5.26) we get

$$\operatorname{Re} R_n(e^{-2\delta}) \ge e^{\alpha\kappa n} \min_{j>0} Q(j) \mathscr{C}(7),$$

for some positive constant $\mathscr{C}(7)$. Consequently, $\lim X_n\{\exp(-2\delta)\} \ge \exp\{\alpha\kappa\} > 1$ for any δ sufficiently small. This contradicts (5.3). Hence, Theorem 1.4 is proved. ∎

6. Proof of Theorem 1.6

Let

$$\Pi_{n,m}(u) := \pi_{n,m}\left(\frac{ua_n}{a_{n+1}}\right) \Big/ \left(a_n(ua_n/a_{n+1})^n\right).$$

From Lemma 3.7, we obtain

$$\Pi_{n,m+1}(u) - \Pi_{n,m}(u) = \frac{(-1)^m}{a_n} \frac{D(n+1,m+1)}{D(n,m)} \cdot \left(\frac{ua_n}{a_{n+1}}\right)^{m+1} \times \frac{1}{Q_{n,m}(ua_n/a_{n+1}) Q_{n,m+1}(ua_n/a_{n+1})}.$$

Now utilizing Lemma 3.1 we see that except at the zeros of B_m, B_{m+1} there holds

$$\lim_{n\to\infty} \{\Pi_{n,m+1}(u) - \Pi_{n,m}(u)\} = \frac{(-1)^m u^{m+1}}{B_m(u) B_{m+1}(u)} \prod_{j=1}^{m} (1-\eta^j) \quad (6.1)$$

(we set $\prod_{j=1}^{m}(1-\eta^j) = 1$ for $m = 0$). We remark that (6.1) is also valid in the case when η is a root of unity, but $\eta, \ldots, \eta^m \neq 1$.

On writing (recall (2.2))

$$\Pi_{n,m}(u) - Z_n(u) = \sum_{k=0}^{m-1} (\Pi_{n,k+1} - \Pi_{n,k})(u),$$

we obtain (1.9) from (6.1). ∎

Proof of Corollary 1.7. This follows immediately on multiplying (1.9) by $Q_{n,m}(ua_n/a_{n+1})$ and applying Lemma 3.1 and Theorem 1.1. ∎

Appendix

Proof of Lemma 3.3. Let ε be an arbitrary positive number. By (1.6), for any integer $s \geq 0$, there is a positive integer n_s such that

$$|\eta_n - 1 - c_1/n - c_2/n^2 - \cdots - c_s/n^s| < \varepsilon/n^s \quad (A.1)$$

for every $n \geq n_s$; we assume that $n_0 \leq n_1 \leq \cdots \leq n_s$. Obviously there exists, for $s \geq 0$, a nonnegative constant $C'(s)$ such that for $n \geq n_s$

$$n^{s+1}|\eta_n - 1 - c_1/n - \cdots - c_s/n^s| \leq C'(s). \quad (A.2)$$

($C'(s) > 0$, if $c_{s+1} \neq 0$; if $c_{s+1} = 0$, then we set $C'(s) := \varepsilon$.)

For $z \neq 0$, let $\log z$ be the principal logarithmic function, i.e., $\log z := \ln|z| + i \operatorname{Arg} z$, $-\pi < \operatorname{Arg} z \leq \pi$. Then, for $|w| < 1$, there holds

$$\log(1+w) = \sum_{k=1}^{\infty} (-1)^{k-1} w^k/k.$$

Before proving Lemma 3.3, we shall first show that for each N and $j < n/(N+1)$ there holds

$$\log \prod_{l=1}^{j} \eta_{n-l} = \sum_{l=1}^{j} \log \eta_{n-l} = \sum_{l=1}^{N} jp_{s-1}(j)/n^s + \mathscr{M}_{N+1}(j,n), \quad \text{(A.3)}$$

where p_{s-1} are polynomials of degree not exceeding $s-1, s=1,\ldots, N-1$, which depend only on c_i, $i=1,\ldots,N$;

$$\mathscr{M}_N(j,n) = \mathscr{M}_{N+1}(j,n) + jp_{N-1}(j)/n^N \quad \text{(A.4)}$$

and

$$\left| n^{N+1}\mathscr{M}_{N+1}(j,n) \right| < c_1(N+1)j^{N+1}, \qquad \text{as } n \to \infty, \quad \text{(A.5)}$$

where $\mathbf{c}_1(N+1)$ is a constant that depends on N.

Indeed, let $j < n/(N+1)$. For $n > 2n_{N+1}$ we may write

$$\sum_{l=1}^{j} \log \eta_{n-l} = \sum_{l=1}^{j} \sum_{s=1}^{N} d_s/(n-l)^s + d_{N+1} \sum_{l=1}^{j} 1/(n-l)^{N+1} + \sum_{l=1}^{j} o\left(1/(n-l)^{N+1}\right)$$

(obviously, $d_1 = c_1$). We rewrite the last equation in the form

$$\sum_{l=1}^{j} \log \eta_{n-l} = \sum_{s=1}^{N} q_s(j)/n^2 + \mathscr{M}_{N+1}(j,n), \quad \text{(A.6)}$$

where

$$\mathscr{M}_{N+1}(j,n) = \sum_{l=1}^{j} \sum_{s=1}^{N} d_s/n^s \cdot \sum_{k=N-s+1}^{\infty} (-l/n)^k \binom{-s}{k} + d_{N+1} \sum_{l=1}^{j} 1/(n-l)^{N+1} + \sum_{l=1}^{j} o\left(1/(n-l)^{N+1}\right) \quad \text{(A.7)}$$

and

$$q_s(j) = \sum_{k=0}^{s-1} d_{s-k} \binom{-s+k}{k} (-1)^k \sum_{l=1}^{j} l^k.$$

We see that for each s, $s \le N$, $q_s(j)$ is a polynomial of degree not exceeding s, since, as it is known, for each fixed nonnegative integer k the sum $q(k,j) := \sum_{l=1}^{j} l^k$ is a polynomial of degree exactly $k+1$ and $q(k,j) = j \cdot q'(k,j)$, where $q'(k,j)$ is a polynomial of degree k. Therefore, q_s may be written in the form $q_s(j) = j \cdot p_{s-1}(j)$, where $\deg p_{s-1} \le s - 1$. We notice also that $q_s(j)$ depends only on $d_1, \ldots, d_s$. This, as well as the construction of $\mathscr{M}_{N+1}(j,n)$ prove (A.3) and (A.4). Also observe that $p_0 = c_1$.

Let us now consider $n^{N+1}\mathscr{M}_{N+1}(j,n)$ for $n > 2n_{N+1}$ and $j < n/(N+1)$. From (A.1) we get

$$\left|n^{N+1}\mathscr{M}_{N+1}(j,n)\right| \le j \cdot e \cdot (\varepsilon + |d_{N+1}|) + \mathscr{L}, \tag{A.8}$$

where

$$\begin{aligned}\mathscr{L} &\le \max_{1\le s\le N} |d_s| \sum_{l=1}^{j} \sum_{s=1}^{N} n^{N+1-s}\left\{\sum_{k=N+1-s}^{\infty}\binom{s+k-1}{k}(l/n)^k\right\}\\ &\le j \max_{1\le s\le N}|d_s| \sum_{s=1}^{N}\{j^{N+1-s}/(s-1)!\}\\ &\quad\cdot \sum_{k=0}^{\infty}\{(k+N)!/(k+N+1-s)!\}\cdot(j/n)^k.\end{aligned}$$

It is easy to check that for $j < n/(N+1)$ the inequality

$$\mathscr{L} < C_1^{(1)}(N+1)j^{N+1}$$

holds, where $C_1^{(1)}(N+1)$ is a positive constant. Combining this result and (A.8), we get (A.5).

We now turn to the proof of the lemma. Let $N \in \mathbf{N}$ and ε be fixed, $n > 2n_{N+1}$, and $j < n/(N+1)$. For $\prod_{l=1}^{j}\eta_{n-l}$ we have

$$\begin{aligned}\prod_{l=1}^{j}\eta_{n-l} &= 1 + \sum_{k=1}^{N}(1/k!)\left\{\sum_{s=1}^{N-k+1} j\cdot p_{s-1}(j)/n^s + \mathscr{M}_{N+2-k}(j,n)\right\}^k\\ &\quad+ \sum_{k=N+1}^{\infty}(1/k!)\left(\log\prod_{l=1}^{j}\eta_{n-l}\right)^k.\end{aligned}$$

We rewrite $\prod_{l=1}^{j}\eta_{n-l}$ in the form

$$\prod_{l=1}^{j}\eta_{n-l} = 1 + \sum_{s=1}^{N} j\cdot\mathscr{P}_{s-1}(j)/n^s + M_{N+1}(j,n),$$

where $M_{N+1}(j,n)$ is given by

$$
\begin{aligned}
M_{N+1}&(j,n)\\
&= \mathscr{M}_{N+1}(j,n) + \left\{ \sum_{k=2}^{N} (1/k!) \cdot \left(\sum_{s=1}^{N-k} j \cdot p_{s-1}(j)/n^s \right)^k \right\}_{(-N-1)}\\
&\quad + \sum_{k=2}^{N} (1/k!) \sum_{r=0}^{k-1} \binom{k}{r} \left(\sum_{s=1}^{N-k+1} j \cdot p_{s-1}(j)/n^s \right)^r\\
&\quad \cdot \left(\mathscr{M}_{N-k+2}(j,n) \right)^{k-r}\\
&\quad + \sum_{k=N+1}^{\infty} (1/k!) \cdot \left(c_1 j/n + \mathscr{M}_2(j,n) \right)^k
\end{aligned}
\tag{A.9}
$$

and

$$
(\cdots)_{(-N-1)} := \frac{\cdots}{n^{N+1}} + \frac{\cdots}{n^{N+2}} + \cdots .
$$

Since $\deg p_{s-1} \le s-1$, $s = 1, \ldots, N$, it follows that $\mathscr{P}_{s-1}(j)$ are polynomials of degree $\le s-1$, respectively.

We consider $n^{N+1} M_{N+1}(j,n)$ as $n \to \infty$ and $j < n/(N+1)$. For $n^{N+1} \mathscr{M}_{N+1}(j,n)$ we apply (A.5). Denote the last sum in (A.9) by B_N. In view of (A.5), we may write

$$
|c_1 j/n + \mathscr{M}_2(j,n)| \le |c_1| j/n + \mathbf{c}_1(2) j^2/n^2 \le (j/n) C_1^{(2)},
$$

so that

$$
\begin{aligned}
|n^{N+1} B_N| &\le j^{N+1} \left(C_1^{(2)} \right)^{N+1} \sum_{k=N+1}^{\infty} (1/k!) \left(j C_1^{(2)}/n \right)^{k-N-1}\\
&\le j^{N+1} C_1^{(3)}(N+1).
\end{aligned}
\tag{A.10}
$$

Now we set

$$
\begin{aligned}
A_N = \sum_{k=2}^{N} (1/k!) \sum_{r=0}^{k-1} \binom{k}{r} &\left(\sum_{s=1}^{N-k+1} j \cdot p_{s-1}(j)/n^s \right)^r\\
&\cdot \left(\mathscr{M}_{N-k+2}(j,n) \right)^{k-r}
\end{aligned}
$$

and consider $n^{N+1}A_N$. For each $r = 0, \ldots, k-1$, we easily obtain that

$$\left(\sum_{s=1}^{N-k+1} j \cdot p_{s-1}(j)/n^s\right)^r$$

$$= (j/n)^r \sum_{\substack{l_1 + \cdots + l_{N-k+1} = r \\ 0 \le l_i \le r}} \mathscr{A}_{l_1, \ldots, N-k+1} \cdot \prod_{i=1}^{N-k+1} \{p_{s-1}(j)/n^{s-1}\}^{l_s},$$

where $\mathscr{A}_{l_1, \ldots, N-k+1}$ are the corresponding coefficients, so that we may write

$$\left(\sum_{s=1}^{N-k+1} j \cdot p_{s-1}(j)/n^s\right)^r \le C_1^{(4)}(N+1) j^r/n^r.$$

From here and from (A.5) we get

$$|n^{N+1}A_N| \le n^{N+1} \sum_{k=2}^{N} (1/k!) \sum_{r=0}^{k-1} \binom{k}{r} (c_1(N-k+2))^{k-r}$$

$$\cdot (j/n)^{(N-k+2)(k-r)+r} (C_1^{(4)}(N+1))$$

$$\le C_1^{(5)}(N+1) \cdot j^{N+1} \cdot \sum_{k=2}^{N} (1/k!)$$

$$\times \sum_{r=0}^{k-1} (j/n)^{(N-k+2)(k-r)+r-N-1}.$$

Finally, we obtain

$$|n^{N+1}A_N| \le C_1^{(7)}(N+1) \cdot j^{N+1}, \tag{A.11}$$

since, as is easy to verify, $(N-k+2)(k-r) + r \ge N+1$.

Using an analogous argument, it can be shown that

$$n^{N+1}\left\{\sum_{k=2}^{N} (1/k!) \cdot \left(\sum_{s=1}^{N-k} j \cdot p_{s-1}(j)/n^2\right)^k\right\}_{(-N-1)} \le C_1^{(8)}(N+1) j^{N+1}.$$

Combining this result, (A.10), (A.11), and (A.5), we obtain (3.5).

Hereafter we shall write M_N instead of M_{N+1} and shall prove (3.6) for $N > 1$.

The proof of (3.6) is based on properties of $\nabla^p(1/n^m)$, $m, p \in \mathbf{N}$, $n > p$. Set

$$\phi_{m,p}(n) = \nabla^p\left(\frac{1}{n^m}\right).$$

From (3.2) it follows that

$$\left|\phi_{m,p}(n)\right| \leq m(m+1)\cdots(m+p-1)/(n-p)^{m+p}. \quad \text{(A.12)}$$

Now, from (A.7) we have

$$\begin{aligned}\nabla^p \mathscr{M}_N(j,n) &= \sum_{l=1}^{j}\sum_{s=1}^{N-1} d_s \sum_{k=N-s}^{\infty}\binom{-s}{k}\nabla^p(1/n^{s+k})(-l)^k \\ &+ \sum_{l=1}^{j}\sum_{s=N}^{N+p} d_s \sum_{k=0}^{\infty}\binom{-s}{k}\nabla^p(1/n^{s+k})(-l)^k \\ &+ \sum_{l=1}^{j}\nabla^p o\left\{1/(n-l)^{N+p}\right\} = I_1 + I_2 + I_3. \quad \text{(A.13)}\end{aligned}$$

Let ε be a positive constant, $n - p > 2n_{N+p}$, and $j < (n-p)/(N+2p)$. Applying (A.12), we obtain for I_1

$$\begin{aligned}|I_1| &\leq j\cdot \max_{1\leq s\leq N-1}|d_s| \sum_{s=1}^{N-1}\sum_{k=N-s} j^k \\ &\quad \cdot \frac{(s+k-1)!}{k!(s-1)!}\left\{\prod_{i=0}^{p-1}(k+s+i)\right\}/(n-p)^{k+s+p} \\ &\leq j\cdot \max_{1\leq s\leq N-1}|d_s| \sum_{s=1}^{N-1}\{j/(n-p)\}^{N-s}\left\{1/(n-p)^{s+p}\right\} \\ &\quad \cdot \sum_{k=N-s}^{\infty}\{j/(n-p)\}^{k-N+s}\frac{(k+s+p-1)!}{k!(s-1)!}.\end{aligned}$$

Arguing as in the proof of (A.5), we establish that

$$\left|\sum_{k=N-s}^{\infty}\{j/(n-p)\}^{k-N+s}\frac{(k+s+p-1)!}{k!(s-1)!}\right| \leq \mathbf{c}_1^{(1)}(N,p)$$

for $s = 1, \ldots, N-1$. Consequently

$$|I_1| \leq \mathscr{C}_2'(N,p)\cdot j^N/(n-p)^{N+p}. \quad \text{(A.14)}$$

In the same way we get (for $j < (n-p)/(N+2p)$)

$$|I_2| \leq \mathbf{c}_1^{(2)}(N,p) \cdot j/(n-p)^{N+1+p}. \tag{A.15}$$

For I_3 we have, from (A.2) (for $n - p > 2n_{N+p}$)

$$\begin{aligned} n^{N+p}|I_3| &\leq \sum_{k=0}^{p} \sum_{l=1}^{j} \binom{p}{k} \left\{ n^{N+p}/(n-l-k)^{N+p} \right\} \\ &\quad \cdot \left\{ (n-l-k)^{N+p} o(1/(n-l-k))^{N+p} \right\} \\ &\leq \varepsilon j \mathbf{c}_1^{(3)}(N,p). \end{aligned}$$

Combining this result, (A.14), and (A.15), we get from (A.13)

$$\left| n^{N+p} \nabla^p \mathscr{M}_N(j,n) \right| \leq j^N \cdot \mathbf{c}_1^{(4)}(N,p), \qquad n \to \infty \tag{A.16}$$

for $j < (n-p)/(N+2p)$ and a constant $\mathbf{c}_1^{(4)}(N,p)$ that does not depend on j and n. Finally, using (A.9) and (A.12) for $1, 2, \ldots, N$, (3.3), and the fact that $\deg p_s \leq s$, we arrive at (3.6). ∎

Proof of Lemma 3.6. We have

$$\begin{aligned} \nabla^p\{n \nabla M_N(j,n)\} &= \sum_{k=0}^{p} (-1)^k \binom{p}{k} (n-k) \nabla M_N(j, n-k) \\ &= n \sum_{k=0}^{p} (-1)^k \binom{p}{k} \nabla M_N(j, n-k) \\ &\quad - p \sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k} \nabla M_N(j, n-1-k) \\ &= n \nabla^{p+1} M_N(j,n) - p \nabla^p M_N(j, n-1), \end{aligned}$$

so that

$$\begin{aligned} \left| n^{N+p} \nabla^p \{n \nabla M_N(j,n)\} \right| &= n^{N+p+1} \nabla^{p+1} M_N(j,n) \\ &\quad - p n^{N+p} \nabla^p M_N(j, n-1). \end{aligned}$$

Thus Lemma 3.3 furnishes the desired estimate for $j < (n-1-p)/N + 2(p+1))$. ∎

Proof of Lemma 3.8. By definition,

$$f(z) Q_{n,m+1}(z) - P_{n,m+1}(z) = \alpha_{n,m+1} \cdot z^{n+m+2} + \cdots$$

and

$$f(z)Q_{n,m}(z) - P_{n,m}(z) = \alpha_{n,m} \cdot z^{n+m+1} + \cdots .$$

From these relations, we get

$$(Q_{n,m} \cdot P_{n,m+1} - Q_{n,m+1} \cdot P_{n,m})(z) = \alpha_{n,m} \cdot z^{n+m+1}.$$

Utilizing (1.5) we determine that

$$\alpha_{n,m} = (-1)^m D(n+1, m+1)/D(n,m),$$

and (3.11) follows. ∎

Proofs of (4.5)–(4.8). We shall establish the statements (4.5)–(4.8) by induction on m. Consider first the case when $m = 1$. Let ε be a positive number, $N \in \mathbf{N}$, $N \geq 2$, be fixed. From (1.6) we have

$$-c_1/\{n(1-\eta_n)\} = 1 + \sum_{i=1}^{N} g_i/n^i + o(1/n^N), \qquad n \to \infty, \quad \text{(A.17)}$$

where

$$g_1 = -c_2/c_1. \qquad \text{(A.18)}$$

Thus for n sufficiently large, say $n > n_N$, there holds

$$\left| n^N \left\{ c_1/n(1-\eta_n) + 1 + \sum_{i=1}^{N} g_i/n^i \right\} \right| \leq \varepsilon.$$

We assume hereafter that $n > 2n_N$.

For $m = 1$ and $j \leq n - 1$, we have from (4.3)

$$A_{n,j,1} = \left(1 - \prod_{i=0}^{j} \eta_{n-i}\right) / (1 - \eta_n).$$

From (4.11) and Remark 1 of Section 3 we obtain for $N \geq 2$

$$A_{n,j,1} = j + 1 + \sum_{s=1}^{N-1} T_{1,s}(j)/n^s + \mathcal{F}_{N,1}(j,n), \qquad \text{(A.19)}$$

where, in view of (4.12),

$$T_{1,1}(j) = j(j+1)(1+c_1)/2,$$

$$T_{1,s}(j) = \sum_{s_1=0}^{s} (j+1)\mathscr{C}_{s_1}(j) g_{s-s_1}/c_1, \qquad s = 2, \ldots, N-1 \quad \text{(A.20)}$$

($g_0 := 1$), and

$$\mathcal{F}_{N,1}(j,n) = o(1/n^N)\left\{j+1+\sum_{s=1}^{N-1}\frac{(j+1)\mathcal{Q}_{s_1}(j)}{c_1 n^s} + n\mathcal{N}_{N+1}/c_1\right\}$$

$$+ n\mathcal{N}_{N+1}(j,n)\left\{1+\sum_{i=1}^{N} g_i/n^i\right\}/c_1$$

$$+ \sum_{s=1}^{N}(g_s/n^s)\sum_{s_1=N-s}^{N-1}(j+1)\mathcal{Q}_{s_1}(j)/n^{s_1}. \qquad \text{(A.21)}$$

From this we see that $\deg T_{1,s}(j) \le s+1$, $s = 1, \ldots, N-1$. In particular, for $N = 1$, we have

$$\mathcal{F}_{1,1}(j,n) = (j+1)(g_1/n + o(1/n))$$
$$+ n\mathcal{N}_2(j,n)\{1 + g_1/n + o(1/n)\}/c_1.$$

Now consider $n^N\mathcal{F}_{N,1}(j,n)$ for $j < n/(N+2)$. Since for such numbers j the inequality $j+1 < (n+1)/(N+1)$ holds for every n sufficiently large, we may apply (3.8) of Remark 1., Combining (A.21) and (A.2), we obtain for $j < n/(N+2)$

$$|n^N\mathcal{F}_{N,1}(j,n)| \le \varepsilon(j+1+jC_3(2)/n) + C_3(N+1)j^{N+1}O(1)$$
$$+ \sum_{i=1}^{N}|g_s|^{(j+1)}\sum_{s_1=N-s}^{N-1}|\mathcal{Q}_{s_1}(j)|$$
$$\le c(1,N)j^{N+1}$$

for a suitable positive constant $c(1, N)$.

Now let p be fixed, $p \in \mathbf{N}$, and $j < n/(N+2+2p)$. In this case, we also have $j+1 < (n-1-p)/(N+1+2p)$ for n sufficiently large, so that (3.9), for $n > n_{N+p}$, and Lemma 3.6 are applicable with respect to $n^{N+p} \cdot \nabla^p\{n\mathcal{N}_{N+1}(j,n)\}$. Now, using for $n > n_{N+p}$ the representation

$$o(1/n^N) = \sum_{s=N}^{N+p} g_s/n^s + o(1/n^{N+p}),$$

(A.21), and Lemmas 3.2, 3.4, and 3.5, we get inequality (4.7) for $m = 1$.

Now consider $A_{n,j,1}$ for $j \geq in/3$ and for $n > 3n_0$. For $j \leq n-1$ we may write from (4.3) that

$$\begin{aligned}|A_{n,j,1}| &= \left|-1 + \prod_{i=0}^{j} \eta_{n-i}\right| / |1-\eta_n| \\ &= \left|1 + \sum_{i=1}^{j} (1-\eta_{n-i}) \prod_{k=0}^{i-1} \eta_{n-i+k}/(1-\eta_n)\right| \\ &\leq 1 + \sum_{i=1}^{n-n_0} \{((n-i)/n)|1-\eta_{n-i}|\}/\{|1-\eta_n|((n-i)/n)\} \\ &\quad + \sum_{i=n-n_0+1}^{j} \{((n-i)/n)|1-\eta_{n-i}|\}/\{|1-\eta_n|((n-i)/n)\}.\end{aligned}$$

Now, from (A.2) of the Appendix, we get

$$\begin{aligned}1 + \frac{C'(0)}{|n(1-\eta_n)|} \cdot \frac{n(n-n_0)}{n_0} + \frac{n \max_{1<k\leq n_0} |1-\eta_k|}{|n(1-\eta_n)|} \\ \leq 1 + \mathscr{C}'(\varepsilon) n^2,\end{aligned} \tag{A.22}$$

where $\mathscr{C}'(\varepsilon)$ is a constant depending only on ε.

Finally, for $A_{n,n,1}$ we obtain

$$A_{n,n,1} = \left(\prod_{i=1}^{n} \eta_i^i\right)/(1-\eta_n),$$

and so

$$|A_{n,n,1}| \leq C''(\varepsilon) 1 |\eta_0|^{n_0(n_0-1)/2} n,$$

where $\eta_0 := \max|\eta_k|$, $k < n_0$, and $C''(\varepsilon)$ is a suitable positive constant. The last inequality and (A.22) yields (4.8). Thus our assertion is proved for $m = 1$.

Suppose now that our hypothesis is true for $1, 2, \ldots, m-1$, $m \geq 2$; namely for any $N \in \mathbf{N}$, and $j < n/(N+3m-4)$

$$\begin{aligned}A_{n,j,m-1} &= \left\{\prod_{i=1}^{m-1} (j+i)\right\}/(m-1)! \\ &\quad + \sum_{s=1}^{N-1} T_{m-1,s}(j)/n^s + \mathscr{F}_{N,m-1}(j,n), \qquad n \to \infty,\end{aligned}$$

where $T_{m-1,s}(j)$ are polynomials of degree not exceeding $s+m-1$, $s = 1,\ldots,N-1$; for any p, $p=0$ or $p \in \mathbf{N}$ and for $j < (n/(N+m+2(m-2+p)))$, there holds

$$\left| n^{N+p} \nabla^p \mathscr{F}_{N,m-1}(j,n) \right| \le c(m-1,N,p) j^{N+m-1}, \qquad n \to \infty. \quad (A.23)$$

Suppose also that

$$|A_{n,j,m-1}| \le c(m-1) n^m, \qquad n \to \infty \quad (A.24)$$

for $j > n/(3m-3)$, where $c(m-1)$ is a suitable positive constant.

Let ε and $N \in \mathbf{N}$ be fixed. Lemma 3.1 yields, for n sufficiently large ($n > n' = n'(\varepsilon)$) the representation

$$\begin{aligned} &D(n,m)^2/(D(n,m+1)D(n,m-1)) \\ &\quad = (-n/mc_1)\left\{1 + \sum_{i=1}^{N} \beta_{i,m}/n^i + o(1/n^N)\right\}. \end{aligned} \quad (A.25)$$

We assume also that $n' > 2n_N$, where n_N is determined by (A.1). Applying Sylvester's identity, we obtain

$$\begin{aligned} &(D(n+1,m)D(n-1,m))/(D(n,m+1)D(n,m-1)) \\ &\quad = (-n/mc_1)\left\{1 + \sum_{i=1}^{N} \beta_{i,m}/n^i + mc_1/n + o(1/n^N)\right\}. \end{aligned} \quad (A.26)$$

On introducing the notation

$$F_{n,m,N} = \sum_{i=1}^{N} \beta_{i,m}/n^i + o(1/n^N)$$

and using (A.25) and (A.26), we can rewrite (4.4) in the form

$$\begin{aligned} A_{n,j,m} &= (-n/mc_1)(1 + F_{n,m,N}) A_{n,j,m-1} \\ &\quad + (n/mc_1)(1 + F_{n,m,N} + mc_1/n) A_{n-1,j,m-1} \prod_{i=1}^{j} \eta_{n-i}. \end{aligned}$$

From this we obtain for $j < n - m$

$$\begin{aligned} A_{n,j,m} &= (1 + F_{n,m,N})(n/mc_1)(A_{n-1,j,m-1} - A_{n,j,m-1}) \\ &\quad + A_{n-1,j,m-1}\left\{(1 + F_{n,m,N})(n/mc_1)\left(\prod_{i=1}^{j}(\eta_{n-i} - 1)\right) \right. \\ &\quad \left. + \prod_{i=1}^{j} \eta_{n-i}\right\}. \end{aligned} \quad (A.27)$$

First we observe that for n sufficiently large and $j < n/(N + 3m - 1)$ it follows from the induction hypothesis that

$$\begin{aligned}(1 + F_{n,m,N})(n/mc_1)(A_{n-1,j,m-1} - A_{n,j,m-1}) \\ = \sum_{s=1}^{N-1} \left\{ \sum_{s_1=1}^{s} T^*_{m-1,s_1}(j)\beta_{s-s_1,m} \right\} / n^s + \mathcal{F}'_{N,m} \\ =: \sum_{s=1}^{N-1} T^{**}_{m,s}(j)/n^s + \mathcal{F}'_{N,m},\end{aligned} \tag{A.28}$$

where

$$T^*_{m-1,s}(j) = \sum_{k=1}^{s} T_{m-1,s-k+1}(j)\binom{-s-k+1}{k}(-1)^k/mc_1, \quad s = 1, \dots, N-1 \tag{A.29}$$

and

$$\begin{aligned}\mathcal{F}'_{N,m} = \left\{ \sum_{s=1}^{N} \{T_{m+1,s}(j)/n^{s-1}\} \sum_{k=N-s+1}^{\infty} \binom{-s}{k}(-1/n)^k \right\} \left\{ \frac{1 + F_{n,m}}{mc_1} \right\} \\ - n\nabla\mathcal{F}_{N+1,m-1}(j,n)\frac{1 + F_{n,m,N}}{mc_1} \\ + \sum_{s=1}^{N-1} \frac{T^*_{m,s}(j)}{n^s} \sum_{s_1=N-s}^{N} \frac{\beta_{s_1,m}}{n^{s_1}} \frac{1}{mc_1}\end{aligned} \tag{A.30}$$

(here $\beta_{0,m} = 1$). Notice that the degree of $T^{**}_{m-1,s}$ does not exceed $m - 1 + s$.

Also, we have for $j = n/(N + m + 1 + 2(m - 1))$, the inequality

$$|n^{N+1}\nabla\mathcal{F}_{N+1,m-1}(j,n)| \le c(m-1, N+1)j^{N+1+m-1}/n.$$

The last inequality easily implies that

$$|n^{N+1}\mathcal{F}'_{N,m}| \le \mathbf{c}'(m,N)j^{N+m-1}. \tag{A.31}$$

Furthermore for any $p \in \mathbf{N}$, we have

$$n^{N+p}\nabla^p\left(n\nabla\mathcal{F}_{N+1,m-1}(j,n)\right) = n^{N+p+1}\nabla^{p+1}\mathcal{F}_{N+1,m-1}(j,n) + pn^{N+p}\nabla^p\mathcal{F}_{N+1,m-1}(j,n-1).$$

Now using (A.30) and the induction hypothesis, applying Lemma 3.2 to $\mathcal{F}'_{N,m}(j,n)$, and keeping in mind that $\deg T_{m-1,s} \le m-1+s$, we obtain

$$\left|n^{N+p}\nabla^p\mathcal{F}'_{N,m}(j,n)\right| \le c''(m,N,p)\cdot j^{N+m}. \tag{A.32}$$

Notice that in (A.32), $j < n/(N+m+1+2(m-1+p))$.

Now set $B_{N,m}(j,n) := (1+F_{n,m,N})(n/mc_1)(\prod_{i=1}^{j}\eta_{n-i}-1)+\prod_{i=1}^{j}\eta_{n-i}$. By virtue of Lemma 3.3, we may write

$$B_{N,m}(j,n) = 1 + j/m + \sum_{s=1}^{N-1}\mathcal{P}^*_{m,s}(j)/n^s + M^*_{N,m}(j,n), \tag{A.33}$$

where

$$\begin{aligned} M^*_{N,m}(j,n) = {} & j\sum_{s=0}^{N-1}\mathcal{P}_s(j)/n^s\left\{\sum_{s_1=N-s}^{N}\beta_{s_1,m}(1/mc_1)\right\} \\ & +(1/mc_1)nM_{N+1}(j,n)(1+F_{n,m,N}) + M_N(j,n) \\ & + o(1/n^N)(1/mc_1)\sum_{s=1}^{N-1} j\mathcal{P}_s(j)/n^s \end{aligned} \tag{A.34}$$

and

$$\mathcal{P}^*_{m,s}(j) = j\mathcal{P}_{s-1}(j) + \sum_{s_1=0}^{s} j\beta_{s-s_1,m}\mathcal{P}_{s_1}(j)(1/mc_1), \qquad s = 1,\ldots,N-1. \tag{A.35}$$

(Here $\beta_{0,m} = 1$ and, as in the previous notation, $\mathcal{P}_0(j) = c_1$.) Notice that $\deg\mathcal{P}^*_{m,s}(j) \le s+1$.

Notice also that Lemmas 3.2, 3.5, and the bound (3.6) are applicable to $M^*_{N,m}(j,n)$ for $j < n/(N+1)$ and for $j < ((n-p)/(N+1+2p))$, respectively. From (A.34) we get, for $p = 0, 1, \ldots,$

$$n^{N+p}\nabla^p M^*_{N,m}(j,n) \le j^{N+1}c^*(m,N,p). \tag{A.36}$$

For $A_{n-1,j,m-1}B_{N,m}(j,n)$ we get from (A.33) and the induction hypothesis,

$$
\begin{aligned}
&A_{n-1,j,m-1}B_{N,m}(j,n)\\
&\quad=\left\{\prod_{i=1}^{m-1}(j+i)/(m-1)!+\sum_{s=1}^{N-1}\frac{T_{m-1,s}(j)}{n^s}\sum_{k=0}^{N-1-s}\binom{-s}{k}\left(-\frac{1}{n}\right)^k\right.\\
&\qquad\left.+\sum_{s=1}^{N-1}\frac{T_{m-1,s}(j)}{n^s}\sum_{k=N-s}^{\infty}\binom{-s}{k}\left(-\frac{1}{n}\right)^k+\mathscr{F}_{N,m-1}(j,n-1)\right\}\\
&\qquad\times\left\{1+j/m+\sum_{s=1}^{N-1}\mathscr{P}^*_{m,s}(j)/n^s+M^*_{N,m}(j,n)\right\}.
\end{aligned}
$$

Introducing polynomials $T^{***}_{m,s}(j)$ we can write the above equation in the form

$$
\begin{aligned}
&A_{n-1,j,m-1}B_{N,m}(j,n)\\
&\quad=\left\{\prod_{i=1}^{m-1}(j+i)/(m-1)!+\sum_{s=1}^{N-1}\frac{T^{***}_{m,s}(j)}{n^s}+\mathscr{F}^*_{N,m}(j,n)\right\}\\
&\qquad\times\left\{1+\frac{j}{m}+\sum_{s=1}^{N-1}\frac{\mathscr{P}^*_{m,s}(j)}{n^s}+M^*_{N,m}(j,n)\right\}. \qquad \text{(A.37)}
\end{aligned}
$$

Notice that $\deg T^{***}_{m,s}(j)\le s+m-1$.

Obviously, inequality (A.23) with respect to $\mathscr{F}_{N,m-1}(j,n-1)$ holds for $j<n/(N+m+1+2(m-1+p))$, $p=0,1,\ldots,$ and for n sufficiently large. From (A.27), (A.28), and (A.37), we finally get

$$
A_{n,j,m}=\prod_{i=1}^{m}(j+i)/m!+\sum_{s=1}^{N-1}T_{m,s}(j)/n^s+\mathscr{F}_{N,m}(j,n),
$$

where

$$
\begin{aligned}
T_{m,s}(j)&=\prod_{i=1}^{m-1}(j+i)\mathscr{P}^*_{m,s}(j)/(m-1)! \qquad \text{(A.38)}\\
&\quad+\sum_{s_1=1}^{s}T^{***}_{m-1,s_1}(j)\mathscr{P}^*_{m,s-s_1}(j)+T^{**}_{m,s}(j),
\end{aligned}
$$

$s = 1, \ldots, N-1$, and

$$\begin{aligned}\mathcal{F}_{N,m}(j,n) = {} & \mathcal{F}'_{N,m}(j,n) \\ & + \mathcal{F}^*_{N,m-1}(j,n-1) \\ & \times \left\{1 + j/m + \sum_{s=1}^{N-1} \mathcal{P}^*_{m,s}(j)/n^s + M^*_{N,m}(j,n)\right\} \\ & + M^*_{N,n}(j,n)\left\{\prod_{i=1}^{m-1}(j+1)/(m-1)! + \sum_{s=1}^{N-1} T^{***}_{m,s}(j)/n^s\right\} \\ & + \sum_{s=1}^{N-1}\{T^{***}_{m,s}(j)/n^s\} \sum_{s_1=N-s}^{N-1} \mathcal{P}^*_{m,s_1}(j)/n^{s_1}. \end{aligned} \tag{A.39}$$

Obviously, the polynomials $T_{m,s}(j)$ do not depend on j and n and $\deg T_{m,s}(j) \leq m + s$. Also, taking $j < n/(N + m + 1 + 2(m - 1 + p))$, $p = 0, 1, \ldots,$ using (A.31), (A.32), (A.36), and the fact that $\deg T_{m-1,s} \leq s + m - 1$, we obtain the required inequalities (4.6) and (4.7) for suitable constants $c(N, m)$ and $c(N, m, p)$.

Next we consider the behavior of $A_{n,j,m}$ for $j \geq n/3m$ as $n \to \infty$. For $n - m > j \geq n/(3m - 3)$, (4.8) follows from the induction hypothesis (A.24) and from (A.27). For $n/3m \leq j < n/(3m - 3)$ we use (A.27), (A.24), (A.23) for $p = 0$ and the induction hypothesis. And finally, for $j > n - m$, (4.8) follows from (4.3). Thus, we have shown that the induction hypothesis is true. ∎

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